Study Guide SF3625
Theory of Partial Differential Equations.

This is not really a study guide. It is an attempt to give a list of the most important definitions and Theorems of the course. To try to write such a list is, though, a fools errand. Everything is interconnected and mentioning one concept, say Sobolev spaces, would immediately require an understanding of other concepts such as the Lebesgue integral, but who could understand the Lebesgue integral set without also knowing what a Lebesgue measure is (?) and understanding measures is pretty much impossible without knowing a fair deal of point-set topoloy which is incomprehensible without an understanding of... you get the point. But here I will single out the parts of the course that I want you to focus on and try to understand. I have also included some information about the exam and a list of the sections in the book that I want you to really study, the sections that I want you to skim, and also some sections that was on the reading-list but we did not cover at all during the lectures and I don't expect you to read.

A summary of the most important parts of the course (in my view):[1]

1. Laplace equation. The Laplace equation is one of the most important partial differential equations. It is also the easiest equation to analyse; therefore much of the PDE theory has been inspired by results first proved for the Laplace equation. Knowledge of the Laplace equation therefore serves as a good foundation for all the more abstract results later in the course. The most important theory for the Laplace equation is:

1. Solving the Poisson equation (Theorem 1 in section 2.2.1)[2]
2. The mean value formula (Theorem 2 in section 2.2.2).
3. The strong maximum principle (Theorem 4 section 2.2.3) and uniqueness of solutions (this is a direct consequence of the maximum principle) (Theorem 5 in section 2.2.3).
4. The definition and of a Green's function and the main theorem for Green's functions (Definition in Section 2.2.4 and Theorem 12 in the same section). One should also have some basic understanding of the Greens function in a half-space and unit ball (say Theorem 14 on the Poisson formula in a half space in Section 2.2.4).

2. The heat and wave equation. The heat and wave equations are model equations for parabolic and hyperbolic partial differential equations and therefore important to study. Simple things, such as the fact that you need an initial data (but cannot specify the final data) in order to get a unique solution for the heat equation but that you need to specify both $u(x,0)$ and $\frac{\partial u(x,0}{\partial t}$ for the wave equation directly carries over to more general parabolic and hyperbolic equations. However, the approaches for the Laplace equation and the heat/wave equations are similar and if you understand the Laplace equation well then it will be easy to pick up the corresponding theory for the heat/wave equations - therefore I will expect that you know the theory for the Laplace equation in more detail.

The most important theory for the heat/wave equation is:

1. Solving the initial value problem for the heat equation (Theorem 1 in section 2.3.1).
2. the Duhamel's principle (Theorem 2 in section 2.3.1), you will also need this for the wave equation.
3. Uniqueness of the heat equation by using energy methods (Theorem 10 in section 2.3.4). Since energy methods are used very often in the weak formulation of PDE you might as well get used to it in the easiest case.
4. It is rather technical (and quite frankly nasty) to derive Kirchoff's formula for the wave equation in two and three dimensions (D'Alembert's formula for $n = 1$ is somewhat nicer). But try to read the construction carefully at least once (section 2.4.1).
5. Read the sections on non-homogeneous solutions of for the wave equation and the energy method proof of uniqueness (section 2.4.2 and 2.4.3). Focus on the similarities in these proofs with the corresponding theorems for the heat equation.

3. Sobolev spaces. Throughout the course I have been repeating that the only thing we do is first year calculus. This is true at least partially true. Much of the theory for PDE can be seen as doing calculus in Banach spaces. The right Banach spaces for studying weak solutions of PDE are the Sobolev spaces. Sobolev spaces have the right generality for us to define derivatives in a way that we recognise while still having many of the nice properties of $\mathbb{R}^n$ (they are in particular complete and weakly compact). It is absolutely necessary to have a basic understanding of Sobolev spaces in order to study weak solutions of PDE.

The most important theory is:

[1] But since I will writhe the exam - my view is supreme in this context!

[2] All my references are to the first edition. But I hope that the sections and theorems have the same numbering in the second edition.

1. The definition of Sobolev spaces (weak derivatives, norms, et.c. in sections 5.2.1 and 5.2.2)

2. That Sobolev spaces are Banach spaces (Theorem 2 in section 5.2.3).

3. Approximation by smooth functions. There are different theorems for this but make sure to understand the local approximation by smooth functions (Theorem 1 in section 5.3.1).

4. Sobolev's and Morrey's inequalities are central (Theorem 1 in section 5.6.1 and Theorem 4 in section 5.6.2).

5. Compactness is one of the most important concepts in analysis so the Rellich-Kondrachev compactness Theorem is absolutely central to the theory of Sobolev spaces (Theorem 1 in section 5.7).

6. You should know the relation between difference quotients and weak derivatives (Theorem 3 section 5.8.2) since this is what is needed to prove higher regularity for weak solutions to PDE.

4. Elliptic equations. The most important theory is:

1. The definition of weak solutions in section 6.1.2. Obviously you should not only memorize the definition. You should try to convince yourself that the definition makes sense and that the very weak assumptions on the weak solution (essentially, we only assume that a solution to a second order PDE only have one derivative!) is very fine tuned in order for us to be able to show existence for a very abstract and complicated set of equations.

2. The Lax-Milgram Theorem (Theorem 1 in section 6.2.1). I do not like the proof and I am inclined to say that it would be much better to include the proof of Riesz representation theorem. But the Lax-Milgram Theorem is the abstract theorem from functional analysis that allows us to show existence of solutions to elliptic PDE - so read it.

3. Energy estimates (Theorem 2 in section 6.2.2). It is more or less always some kind of estimate that allows us to use the functional analysis we need in order to prove existence of solutions.

4. The first existence theorem for weak solutions (Theorem 3 in section 6.2.2).

5. Interior regularity (Theorem 1 in section 6.3.1). The more you study PDE the more you will realize that it is estimates of different kinds that are the most important results. For now we should agree that it is at least comforting to know that the solutions of second order equations has second derivatives in some sense.

6. Hopf's Lemma and the strong maximum principle (Theorem 3 and the lemma in section 6.4.2). The Hopf lemma uses a very common technique in PDE theory: Barriers. So it is nice to see the technique in this course.

5. Parabolic/Hyperbolic PDE. Above I mentioned that much of the theory for Laplace equation carries over directly to the heat/wave equation. The same is true for elliptic and parabolic/hyperbolic equations. Also much of the theory for parabolic equations carries over directly to hyperbolic equations. Therefore it is natural to study elliptic equations with more care than parabolic and parabolic with more care than hyperbolic.[3]

I will only expect you to skim through much of the theory for parabolic/hyperbolic equations. But there are some new techniques for evolution equations that I want you to study. The most important theory is:

1. the definitions of weak solutions (you will find the definitions in sections 7.1.1 and 7.2.1).

2. Galerkin's method to construct approximate solutions (basically section 7.2.1a for the parabolic case; skim through section 7.2.2a with enough care so that you see that the method is the same for hyperbolic equations).

3. You should have an idea of the proof of how to show existence of solutions (Theorem 3 in section 7.1.2b). The main point of the proof is that we may estimate the norm of the approximate solutions and thus pass to the limit. You need, in particular to understand Theorem 2 in section 7.1.2b. The proofs are very similar for hyperbolic equations so do not feel obliged to read them.

4. The weak maximum principle for parabolic equations.

6 Calculus of variations. Calculus of variations is the only part of this course where we show any kind of existence of solutions for non-linear equations. There are 3 main things that I would want you to read carefully:

1. Existence of weak solutions (Theorem 2 in section 8.2.2). But the central part of Theorem 2 is Theorem 1 (lower semi-continuity for convex energies) in the same section. You should know them both.

2. Uniqueness of minimizers (Theorem 3 in section 8.2.2). Note that this theorem is just the same as uniqueness of the minimum of a strictly convex function on $\mathbb{R}$.

[3] This probably true only in a course like this. There is, for instance, a huge theory for hyperbolic equations that does not have any counterpart in the elliptic/parabolic theory. But in this course when we focus on the theory of weak solutions it makes sense to study elliptic equations with more care and then just skim over the parabolic/hyperbolic theory where the theory is more or less the same.

3. That the minimizers satisfies the Euler-Lagrange equations (Theorem 4 in section 8.2.3).

Chapters included in the course.

Technically the course includes all the material on the reading list for the lectures. But in reality I only expect you to skim parts of the text. I will split the text into three kinds of importance: what you have to read, what you should skim through, and what you don't have to read but was on the list.

Again I will refer to sections (believing that the sections agree between editions).

The important parts that you have to read:

1. Chapter 2.2, 2.3.1, 2.3.4, 2.4.1abc, 2.4.2, 2.4.3.
2. Chapter 5.1, 5.2, 5.3.1, 5.6.1, 5.6.2, 5.7, 5.8.2a
3. Chapter 6.1, 6.2.1, 6.2.2, 6.3.1 (skim the higher regularity theorem), 6.4
4. Chapter 7.1.1, 7.1.2, 7.1.4a, 7.2.1, 7.2.2a
5. Chapter 8.1.1, 8.1.2, 8.1.3, 8.2.1, 8.2.2, 8.2.3

The parts that you should have some knowledge of and need to skim through.

1. Chapter 2.3.2, 2.3.3
2. Chapter 5.3.2, 5.6.3, 5.8.1
3. Chapter 7.1.3, 7.2.2b, 7.2.2c, 7.2.3, 7.2.4
4. Chapter 8.3.1

What I did not have time to mention at all during the lectures and will not ask about at the exam.

1. Chapter 2.4.1de
2. Chapter 7.1.4b
3. Chapter 8.1.4, 8.2.4, 8.3.2

Some information about the exam:

Format: The final exam will consist of 4 problems and last for 4 hours. Two of these problems will be taken from your homework sets (or slight variations of homework problems).

When and where: The exam will be on **the 18th of January from 8:00-12:00 in room 3418** (the usual lecture room).

What is expected of you at the exam? Even though I have tried to cut down the reading significantly there is still to much to memorize for an exam. I will try to make the exam rather easy so that if you read all the "important theory" from the list above you will be able to pass. I do not want you to try to memorize it all - that is not worth your time. But I want you to be able to understand the theory when you read it and I will try to write the exam to check that you have spent some time with the course material.