PDE SF3625 Homework 3.

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This is your homework assignments for the third part of the course covering primarily weak solutions and variational calculus.[1] **Do not forget to add your name, email and Swedish personal id number** (if you have one) to your solutions.

You will hand in one exercise (your choice which) from Part 1 and one exercise (again your choice) from Part 2.

Your solutions are due on the 11th of January (before midnight).

Part 1, Weak solutions.

1. The Navier-Stokes equations consists of a system of PDE in a domain $D \times (0,T)$, $D \subset \mathbb{R}^3$,

$$\frac{\partial \mathbf{u}(x,t)}{\partial t} - \Delta \mathbf{u}(x,t) + \mathbf{u}^T \cdot \nabla \mathbf{u}(x,t) - (\nabla p(x,t))^T = \mathbf{f}(x,t) \qquad (1)$$

where $\mathbf{u}$ and $\mathbf{f}$ are vector valued functions:

$$\mathbf{u}(x,t) = \begin{bmatrix} u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{bmatrix} \text{ and } \mathbf{f}(x,t) = \begin{bmatrix} f_1(x,t) \\ f_2(x,t) \\ f_3(x,t) \end{bmatrix},$$

and $\mathbf{u}^T$ denotes the transpose of $\mathbf{u}$. The function $p(x,t)$ is a scalar valued function: "the pressure".[2] Also $\mathrm{div}(\mathbf{u}) = 0$ in the weak sense:

$$\int_D \mathbf{u}^T(x,t) \cdot \nabla g(x) dx = 0 \qquad \text{for a.e. } 0 < t < T \text{ and every } g \in W_0^{1,2}(D). \qquad (2)$$

We say that a function satisfying $\mathbf{u} \in L^2((0,T); W_0^{1,2}(D))$ and $\frac{\partial \mathbf{u}}{\partial t} \in L^2((0,T); H^{-1}(D))$ is a weak solution of the Navier-Stokes equations if $\mathbf{u}$ satisfies (2) and for a.e. $t \in (0,T)$

$$\left\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right\rangle + B[\mathbf{u}, \mathbf{v}, t] = (\mathbf{f}, \mathbf{v}) \qquad (3)$$

and every $\mathbf{v} \in W^{1,2}(D)$ that is divergence free in the sense of (2), and $\mathbf{u}(x,0)$ has some specified values in L^2. Here B is the non-linear form

$$B[\mathbf{u}, \mathbf{v}, t] = \int_D \left(\sum_{i=1}^3 \nabla u_i(x,t) \cdot \nabla v_i(x,t) + \sum_{i,j=1}^3 u_j(x,t) \frac{\partial u_i(x,t)}{\partial x_j} v_i(x,t) \right) dx.$$

[1] If you email your solutions email me a PDF file that is either computer written or a scan of your handwritten solutions. Do not send me photos of your solution since they are usually very difficult to read.

[2] Do not care about the pressure, it is some form of Lagrange multiplier of the restriction that $\mathbf{u}$ is divergence free and will not play any part in our weak formulation.

Assuming that we can find an orthogonal basis of $W_0^{1,2}(D)$ of divergence free vector-valued functions $\{\mathbf{w}_k\}_{k=1}^\infty$ that are orthonormal in $L^2(D)$. Make an argument (don't provide all the details) that we can show existence of weak solutions for the Navier-Stokes equations if we can prove the estimate[3]

$$\|\mathbf{u}\|_{L^2((0,T);W_0^{1,2}(D))}+\left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{L^2((0,T);H^{-1}(D))} \leq \|\mathbf{f}\|_{L^2((0,T);L^2(D))}+\|\mathbf{u}(x,0)\|_{L^2(D)}.$$

2. [Homogenization] Let $a(x) \in L^\infty(\mathbb{R}^n)$ be defined by

$$a(x) = \begin{cases} 2 & 2k < x_1 \leq 2k+1 \text{ for } k \in \mathbb{N} \\ 1 & 2k+1 < x_1 \leq 2k+2 \text{ for } k \in \mathbb{N}. \end{cases}$$

Also let $u_\epsilon(x) \in W_0^{1,2}(D)$, for $\epsilon > 0$, be the solution, in a domain $D \subset \mathbb{R}^n$, of the following PDE

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(a(x/\epsilon)\frac{\partial u_\epsilon(x)}{\partial x_i}\right) = f(x) \in L^2(D). \tag{4}$$

Show that $u_\epsilon \to u_0$ weakly in $W_0^{1,2}(D)$ as $\epsilon \to 0$ (maybe through a subsequence).

Part 2, Calculus of variations.

3. [Convexity in the calculus of variations] Let $F(p) = (p-1)^2(p-2)^2$ and define the energy

$$E(u) = \int_0^1 F\left(\frac{\partial u(x)}{\partial x}\right) dx$$

on the set

$$K = \{u \in W^{1,4}((0,1));\ u(0) = 0 \text{ and } u(1) = 3/2\}.$$

Show that for any function $g \in C^1([0,1])$ such that $1 \leq g'(x) \leq 2$, $g(0) = 0$ and $g(1) = 3/2$ there exists a minimizing sequence $u_j \in K$ satisfying $E(u_j) = 0$ and $u_j \to g$ weakly in $W^{1,4}((0,1))$. Will g be a minimizer?

4. [About coercivity] Given a function $f \in L^\infty(B_1(0))$, $B_1(0) \subset R^n$, we define the energy

$$E(u) = \int_{B_1(0)} \left(|\nabla u(x)|^2 - |u(x)|^p + f(x)u(x)\right) dx$$

on the set $u \in W_0^{1,2}(B_1(0)) \cap L^p(B_1(0))$.

Show that there exists a constant $\hat{p} > 1$ depending only on n such that

[3] The estimate is very easy to prove, try if you want to.

1. If $1 \leq p < \hat{p}$ then the energy E has a unique minimizer.
2. If $\hat{p} < p$ then the energy E does not have any minimizer.

5. In the proof (in Evans) of the theorem stating that if

$$\text{the mapping } p \mapsto L(p,z,x) \text{ is convex}$$

then the energy

$$I[u] = \int_U L(\nabla u, u, x)dx$$

is lower semi-continuous[4] evans actually uses more than convexity. During the lectures we also assumed some continuity in the z variable of L and of $\nabla_p L(p,z,x)$.

Explain where this is needed in Evans proof and show that lower semi-continuity in the z variable is enough.

[4]In the first edition of Evans this is Theorem 1 on page 446 in chapter 8.2 - probably it is in chapter 8.2 in the second edition as well