PDE SF3625 Homework 2.

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This is your homework assignments for the second part of the course covering primarily Sobolev spaces in Evans book.[1] **Do not forget to add your name, email and Swedish personal id number** (if you have one) to your solutions.

You will hand in one exercise (your choice which) from Part 1 and one exercise (again your choice) from Part 2.

Your solutions are due on the 7th of December (before midnight).

Part 1, basic Sobolev Spaces.

1. Let $f \in L^p([0,1])$, $1 \le p < \infty$ and define

$$u(x) = \int_0^x f(t)dt.$$

Prove that $u \in W^{1,p}([0,1])$.

2. Let $f : \mathbb{R} \mapsto \mathbb{R}$ have a weak derivative $f' \in L^p(\mathbb{R})$. Prove that

1. if $p \ge 1$ then f is continuous on $\mathbb{R}$.

   REMARK: *Note that this is actually better than what we get from the Morrey embedding theorem: which does not apply in this case since* $p = n = 1$.

2. If $\lim_{x \to -\infty} f(x) = 0$ and $p = 1$ then f is bounded.

3. If $\lim_{x \to -\infty} f(x) = 0$ and $p > 1$ then f is not necessarily bounded.

Part 2, More advanced exercises Sobolev Spaces.

3. Let $f_j \in L^p(\mathbb{R}^n)$, $j = 1, 2, ...$, and assume that each f_j has weak derivatives $\nabla f_j = (\partial_1 f_j, \partial_2 f_j, ..., \partial_n f_j)$ satisfying the following estimate for each ball $B_1(x) \subset \mathbb{R}^n$

$$\int_{B_1(x)} |\nabla f_j(y)|^2 dy \le C \int_{B_1(x)} |f_j(y)|^2 dy$$

for some constant C (C does not depend on f_j or x).

For which p does it follow that: if f_j converge weakly in $L^p(\mathbb{R}^n)$ to f_0 then f_j converge strongly to f_0 in $L^p(\mathbb{R}^n)$.

[1] If you email your solutions email me a PDF file that is either computer written or a scan of your handwritten solutions. Do not send me photos of your solution since they are usually very difficult to read.

4. Let $u \in C_c^\infty(B_1(0))$, $B_1(0) \subset \mathbb{R}^n$, satisfy (for some constant)

$$\int_{B_1(0)} \nabla u(x) \cdot \nabla v(x) dx \leq C \int_{B_1(0)} u(x) v(x) dx$$

for every $v \in C^\infty(B_1)$. For simplicity we assume that $u \geq 0$.

Prove that for any $q \geq 1$ there is a constant C_q such that

$$\|u\|_{L^q(B_1(0))} \leq C_q \|u\|_{L^2(B_1(0))}. \tag{1}$$

HINT: *Choose $u = v$, use (1) and the Sobolev inequality to show that $u \in L^{2^*}$ with control over the norm $\|u\|_{L^{2^*}}$. Then note that for any $\alpha > 1$*

$$|\nabla u^\alpha \cdot \nabla u| = \frac{4\alpha}{(1+\alpha)^2} \left| \nabla u^{\frac{\alpha+1}{2}} \right|^2. \tag{2}$$

You may therefore apply (1) with $v = u^{2^*-1}$, use (2) and the Sobolev inequality to show that $u^{\frac{2^*-2}{2}} \in L^{2^*}(B_1(0))$. Iterate!