PDE SF3625 Homework 1.

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This is your homework assignments for the first part of the course covering Chapter 2 in Evans book.[1] **Do not forget to add your name, email and Swedish personal id number** (if you have one) to your solutions.

You will hand in one exercise (your choice which) from Part 1 and one exercise (again your choice) from Part 2.

Your solutions are doe on the 18th of October (before midnight).

Part 1, basic analysis.

1. In Evan's book, in the displayed formula after equation (16) in Chapter 2.3 (p.51 of the first edition) it is claimed that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \Psi(y,\epsilon) f(x-y, t-\epsilon) dy = f(x,t).$$

It is also remarked that the limit "is computed as in Theorem 1".

1. Explain why the limit is not exactly the same as any limit in the proof of Theorem 1.
2. Provide a proof for the limit.

2. In the proof of Lemma 1 in Chapter 2.4 it is stated that: "after some computations ...

$$U_{rr}(x;r,t) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B_r(x)} \Delta u dS + \left(\frac{1}{n} - 1\right) \frac{1}{\alpha(n)r^n} \int_{B_r(x)} \Delta u dy.''$$

Prove this!

3. During the lectures we have stated that[2] if $f \in C^2(D)$, $u \in C^2(D \cap C(\overline{D})$ and

$$\begin{aligned} &\Delta u(x) = f(x) && \text{in D} \\ &u(x) = 0 && \text{on } \partial D \end{aligned}$$

then

$$u(x) = \int_D f(y) G(x,y) dy, \tag{1}$$

[1] If you email your solutions email me a PDF file that is either computer written or a scan of your handwritten solutions. Do not send me photos of your solution since they are usually very difficult to read.

[2] See also Theorem 12, chapter 2.2 in Evans book.

where $G(x,y)$ is the Green's function for D. However, we have not really proved the formula: we only stated that the proof involved writing $\Delta_y u(y)$ for $f(y)$ in (1) and integrate by parts twice ("arguing as in Theorem 1 in Chapter 2.2"). In this homework I want you to prove (1) under the above assumptions.

Part 2, PDE Theory.

4. Use the methods in the proof of Theorem 10 in Chapter 2.3 to prove that if

$$\begin{array}{ll} \frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) = 0 & \text{in } D \times (0,T) \\ u(x,t) = 0 & \text{on } \partial D \times (0,T) \\ u(x,0) = g(x) & \text{for } x \in D \end{array}$$

then, for any $s \in (0,T)$,

$$\int_0^s \int_D |\nabla u(x,t)|^2 dxdt + \frac{1}{2}\int_D |u(x,s)|^2 dx = \frac{1}{2}\int_D |g(x)|^2 dx.$$

You may assume that u, g and D are C^∞.

5. Consider the boundary value problem, for some given $g \in C_c^\infty(\mathbb{R}^{n-1})$,

$$\begin{array}{ll} \Delta u(x) = 0 & \text{in } \mathbb{R}^n_+ \\ u(x',0) = g(x') & \text{for } x' \in \mathbb{R}^{n-1}. \end{array} \tag{2}$$

1. Show that we do not have uniqueness of solutions for this problem. That is: prove that there exists two different solutions $u(x)$, $v(x)$, with two continuous derivatives in $\overline{\mathbb{R}^n_+}$, to (2).

2. Show that bounded solutions to (2) are unique.

   HINT: *If $u(x)$ is a sloution to (2) with $g(x) = 0$ then*

   $$\hat{u}(x) = \begin{cases} u(x) & \text{for } x_n \geq 0 \\ u(x', -x_n) & \text{for } x_n < 0 \end{cases}$$

   will be harmonic in $\mathbb{R}^n$.

6. The research community has a certain tendency to exaggerate the novelty of its research. Often papers involves some new equation or slight variation of a know equation with a grandiose claim that their results are new.

It would not be difficult to find examples of trivialities that are presented as novel research in many research journals, but in order not to humiliate any of my colleagues I have made up an example of a typical problem that some researchers would try to pass of as new science (but it would not surprise me at all if some one actually has published the following problem as new research).

The point of this exercise is first and foremost to practise the techniques of this course. But also to make you reflect on the difference between new results that are proved by standard methods and genuinely new methods. Let us get to the problem.

The following problem is, to my knowledge entirely "new" in the sense that I have never seen any articles, books et.c. discussing it. Given a bounded function $f \in C(\mathbb{R}^n)$ we look for functions $u \in C_1^2(\mathbb{R}^n \times (0,T)) \cap C(\mathbb{R}^n \times [0,T))$ and $v \in C_2^2(\mathbb{R}^n \times (0,t)) \cap C_1^0(\mathbb{R}^n \times [0,T))$ such that

$$\begin{array}{ll} \frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ \frac{\partial^2 v(x,t)}{\partial t^2} - \Delta v(x,t) = (u(x,t))^2 & \text{in } \mathbb{R}^n \times (0,T) \\ u(x,0) = f(x) & \text{in } \mathbb{R}^n \\ v(x,0) = \frac{\partial v(x,0)}{\partial t} = 0 & \text{in } \mathbb{R}^n. \end{array}$$

Let us call these equations something fancy (so that we can fool journal-editors and university review boards to believe that we are working on something fundamental) like *"non-linear coupled parabolic-hyperbolic systems of partial differential equations"* (sounds impressive enough!).

Prove the following theorem:

Theorem 1. *The non-linear coupled parabolic-hyperbolic system of partial differential equations exhibits infinite speed of propagation in the sense that there exists functions with compact support $f \in C_c(\mathbb{R}^n)$ such that, for arbitrarily small $t > 0$, the support of neither $u(x,t)$ and $v(x,t)$ are compact subsets of $\mathbb{R}^n$.*

7. Let $\Delta u(x) = f(x)$ in some domain $D \subset \mathbb{R}^n$ where $u, f \in C^\infty(D)$. If $f(x) = 0$ then the mean value formula implies that $u(y)$ equals its average in any ball $B_r(y) \subset D$.

A similar statement should be true even if $f(x) \neq 0$. That is, there shold exist a formula such that, for any ball $B_r(y) \subset D$

$$u(y) = \frac{1}{\alpha(n) r^n} \int_{B_r(y)} u(x)dx + T(f),$$

where $T(f)$ is some expression involving f.

Find such a formula.